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Ionisation waves in solids

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Abstract. It is shown that ionisation waves, familiar in connection with the positive column of a gaseous plasma, can occur in solids. This is demonstrated with a simple model in which a single type of carrier (electrons) impact ionise a deep-level trap in an insulator. Impact ionisation is described by lucky-drift theory taking into account the non-local nature of the process. An analytic theory is presented for linear waves and conditions for the existence and stability of ionisation waves are derived. It is shown that stationary, forward-travelling and backward-travelling waves are all possible, and they occur as a consequence of the non-local nature of the impact ionisation process. It is also shown that the presence of stationary waves in a finite length of sample leads to a current-controlled negative differential resistance (NDR). The end point of the growth of stationary waves in time is shown to be saw tooth waves of unchanged wavelength with an amplitude determined by the occupation of the trap at electrical neutrality; being zero for completely full or completely empty traps and being a maximum when the traps are half filled. The theory is applicable to semiconductors as well as insulators.

1. Introduction

Ionisation waves in the positive column of a weakly ionised plasma have been known for many years. An exhaustive experimental study was made by Stewart (1956) and there has been much theory (see, for example, Von Engel (1965) and more recently Grabec (1974)). Recently, ionisation waves were discovered in a numerical simulation of hot-electron transport in ZnS (Ridley and El-Ela 1989) and were seen to arise as a consequence of the non-local nature of the impact ionisation process. As far as the author is aware this was the first intimation that ionisation waves could exist in solids. Because this demonstration by Ridley and El-Ela was numerical it was not clear what the general conditions for the existence of such waves were. This paper is an attempt to illuminate the properties further by presenting a linear theory of ionisation waves.

Since the basic physics of these waves resides in the non-local nature of ionisation we outline in section 2 an approach based on lucky-drift theory (Ridley 1983, 1987, Burt 1985), in which the central concept of the ionisation length, L, is introduced. Section 3 contains the linear theory of ionisation waves with the assumption that the ionisation rate at a point x depends upon the average field over the distance between $x - L$ and x, and the carrier density at $x - L$. In this way we introduce the non-local element. In section 4 the general dispersion relations of linear theory are solved for stationary waves in an insulator, and it is shown that stationary waves exist only for a specific set of wavelengths. For other wavelengths the waves are propagating, either forwards (in the

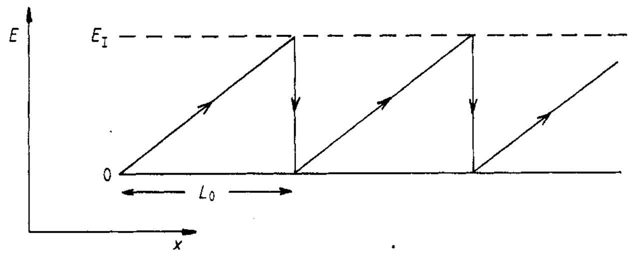

Figure 1. Collisionless electron trajectories.

direction of electron drift) or backwards, and we deal with these in section 5. Whether propagating or stationary, the waves share the same expression for the growth constant. One interesting property of stationary waves in a finite sample is that they can give rise to a current-controlled negative differential resistance (NDR). Limitations to growth imposed by trap occupancy are shown in section 6 to lead to saw-tooth waves.

2. Lucky-drift theory of impact ionisation

The simplest picture of impact ionisation is of an electron which is accelerated by the prevailing electric field to the point when its energy equals the ionisation energy E_I and impact ionisation takes place (figure 1). If the electron starts with zero energy and the field is $\mathscr{E}$ the distance travelled, L_0, is simply

$$L_0 = E_I/e\mathscr{E}. \quad (1)$$

After ionisation the electron has substantially zero energy and so has the impacted electron. A second bout of acceleration over a distance L results in a second plane of ionisation, and so on. Thus impact ionisation has the intrinsic property of appearing as a wave pattern in space with the characteristic wavelength given by L_0.

There are several factors which blur such ionisation waves.

(i) Spread in the starting positions. In the limit of a uniform distribution of electrons in space, waves as conceived above disappear utterly.

(ii) Soft threshold. When the electron energy reaches E_I the probability per unit distance of ionising becomes finite but usually less than unity. On average the electron will travel a further distance ΔL before ionising, but there will be statistical fluctuations which will blur the ionisation plane.

(iii) Elastic collisions. In many systems the chance of the electron avoiding a collision is negligible. Collisions do two things: they randomise momentum and, if inelastic, they remove energy from the electron. Randomising momentum means that the electron can find itself sometimes travelling against the field. When the collisions are frequent the average velocity of the electron changes from being ballistic to being determined by drift. Provided the collisions are elastic the characteristic length is unchanged since this determines the potential difference required for ionisation, so spatial oscillations are still expected. Thus scattering is not important, provided only that the collisions are elastic.

(iv) Inelastic collisions (see figure 2). In order to reach the ionisation threshold in the distance L the electron must avoid an energy losing collision. Following lucky-drift

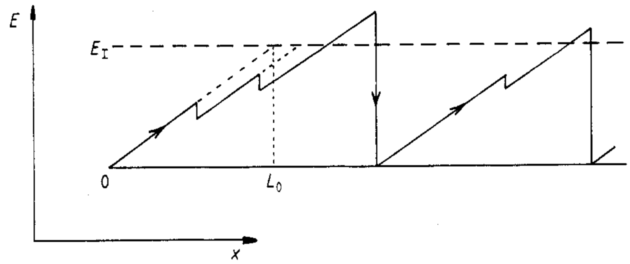

Figure 2. Electron trajectories in the presence of energy-relaxing collisions and a soft threshold.

theory we artificially separate collisions into two distinct categories, those which only relax momentum and those which only relax energy. If the latter occur with a characteristic time constant τ_E the probability of avoiding a collision in time t is

$$P(t) = \exp\left(-\int_0^t \frac{dt}{\tau_E}\right). \tag{2}$$

If, as is usual, momentum relaxation is much more rapid than energy relaxation then dt can be replaced by dx/v'_d where v'_d is the lucky-drift velocity, whence

$$P(L_0) = \exp\left(-\int_0^{L_0} \frac{dx}{v'_d\tau_E}\right) = e^{-L_0/L_E} \tag{3}$$

where $L_E = v'_d\tau_E$ is regarded as energy (and hence spatially) independent. Thus the ionisation coefficient (probability per unit length) is

$$\alpha = (1/L_0)\, e^{-L_0/L_E}. \tag{4}$$

Impact ionisation now occurs at a rate determined by a drifting electron which is lucky enough to avoid an energy-relaxing collision. In solids where energy relaxation is principally via the emission of high-frequency phonons L_E is indeed independent of energy for a parabolic energy band structure, and is simply proportional to the electric field (Ridley 1983), namely

$$L_E = \{[2n(\omega) + 1]\lambda^2/2\hbar\omega\}\, e\mathscr{E} \tag{5a}$$

$$\alpha = (e\mathscr{E}/E_I)\exp[-2r(E_I/e\mathscr{E}\lambda)^2] \tag{5b}$$

where $r = \hbar\omega/[2n(\omega) + 1]E_I$, $\hbar\omega$ = phonon energy, $n(\omega)$ = Bose–Einstein factor, and λ is the mean-free path for collisions. Lucky-drift theory can be elaborated by taking into account ballistic contributions, contributions from the background hot-electron gas (initial electron energy greater than zero), and including a soft threshold (Ridley 1983, 1987). It has been shown to agree extremely well with numerical solutions based on transport theory and a Monte Carlo simulation. Thus, in all situations where motion is dominated by collisions, impact ionisation arises from a lucky drifting electron. (Note that estimates of the impact ionisation rate based on a Maxwell–Boltzmann distribution or a phonon-induced Gaussian distribution are known to be in error. While these distributions correctly account for average quantities they cannot account properly for improbable events such as the lucky-drift electron.)

Let us look at the effect of these factors on simple ionisation waves more quantitatively. A uniform spread of electrons to begin with is at first sight bad for waves, but we shall return to this problem later to ask how stable a uniform state is. The other factors introduce characteristic lengths which must be compared with L_0.

Impact ionisation rates near threshold can be expressed as follows:

$$\frac{1}{\tau_i} = \frac{p}{\tau_E}\left(\frac{E - E_I}{E_I}\right)^s \tag{6}$$

where p is the soft-threshold factor, τ_E is the energy relaxation time, and $s = 2$ for band-to-band impact ionisation (Keldysh 1960) and $s = 1$ for impurity ionisation (Ridley and El-Ela 1989, Woods 1987). The soft-threshold factor in well-investigated semiconductors is about unity for the band-to-band process (Ridley 1987). For an impurity, p will be proportional to the density of trapped electrons, Since $p \simeq 1$ for the band-to-band process where the effective density of ionisable electrons at the top of the valence band is about $10^{21}\,\mathrm{cm}^{-3}$ in semiconductors we can estimate that for impurities $p \simeq n_T/10^{21}$, where n_T is the density of ionisable electrons in units of cm^{-3}. (This turns out to be equivalent of assuming an ionisation cross-section of order $10^{-16}\mathrm{cm}^{-2}$.)

In all cases, except those where the field is very high, the overshoot in energy is determined by the energy relaxation rate rather than the impact ionisation rate. The length of prime importance here is L_E, which is proportional to the field, and the effective length which the electron has to travel becomes

$$L = L_0 + L_E \tag{7}$$

with an uncertainty also of order L_E. Thus the softness of the threshold pushes up the effective magnitude of the threshold energy and demands that the electron travel on average an energy relaxation length further. Ionisation planes are well-defined provided $L_E \ll L_0$, which will be true at low fields.

When nearly all the collisions are inelastic, which is the relastic case, lucky-drift theory points out that impact ionisation is effected principally by these lucky electrons which collide very infrequently, though they usually collide at least once (and hence more in the drift mode rather than in the ballistic mode). Usually the collisions are with high-frequency phonons, each collision resulting in a loss of energy of, on average, $\hbar\omega/[2n(\omega) + 1]$ where $\hbar\omega$ is the phonon energy. Thus if the average loss of energy per collision is small compared with the threshold energy for ionisation the extra distance on electron drifts to make up for energy losses will also be small compared with L_0. In other words we do not expect the existence of weakly inelastic collisions to affect the possibility of ionisation waves materially.

We conclude from all of this that although sharp ionisation planes cannot occur in the real circumstance of solid state impact ionisation, a more diffuse ionisation wave is to be expected when there exists a well-defined starting point for acceleration. In the uniform case, where every point is a starting point, no waves appear to be possible. An analysis of the stability of the uniform case, however, shows that waves do occur, and to this analysis we now turn. The basic requirement is simply that impact ionisation is a non-local effect.

3. Linear theory of ionisation waves in a solid

We consider the case of a semiconductor or insulator containing a population of a single impurity which gives rise to a localised level in the forbidden gap. We neglect thermal processes and take into account only capture and impact ionisation processes involving electrons. We shall build into our theory from the start the basic attributes of the impact ionisation process as envisaged by lucky-drift theory, namely, that the rate at x is determined by (*a*) the electron density at $x - L$ and (*b*) the average electric field between x and $x - L$. The equation determining the rate of change of the trapped electron population is then,

$$\mathrm{d}n_\mathrm{T}/\mathrm{d}t = c(N_\mathrm{T} - n_\mathrm{T})n - e_\mathrm{i}n'n_\mathrm{T} \tag{8}$$

where n_T is the trapped electron concentration, N_T is the total trap concentration, n and n' are the electron concentrations in the conduction band at x and $x - L$ respectively, c is the volume capture coefficient and e_i is the volume impact ionisation coefficient dependent on the average electric field over the distance $x - L$ to x. We will need Gauss's equation:

$$\mathrm{d}\mathscr{E}/\mathrm{d}x = (e/\varepsilon_\mathrm{s})(n + n_\mathrm{T} - n_0 - n_{\mathrm{T}0}) \tag{9}$$

where $\mathscr{E}$ is the electric field (in the x-direction), ε_s is the static permittivity of the solid and the subscript zero denotes the concentrations for electrical neutrality, and we are limiting our attention to variations along the field direction. Finally we require the charge conservation equation,

$$\mathrm{d}\rho/\mathrm{d}t = -\mathrm{d}j/\mathrm{d}x \tag{10}$$

where ρ is the space charge density and j is the current density, given by

$$j = env_\mathrm{d} - eD\,\mathrm{d}n/\mathrm{d}x \tag{11}$$

where v_d is the drift velocity and D is the diffusion coefficient. (Note that v_d here refers to the entire free-electron population whereas in the previous section the primed symbol was used to refer to lucky drift. They should not be confused.)

In the uniform steady state $n = n' = n_0$, $n_\mathrm{T} = n_{\mathrm{T}0}$, where

$$n_{\mathrm{T}0} = cN_\mathrm{T}/(e_\mathrm{i} + c). \tag{12}$$

Also, $\rho = 0$ and $j = en_0v_\mathrm{d}$.

In order to examine the stability of this solution we perturb the quantities $\mathscr{E} \rightarrow \mathscr{E}_0 + \delta\mathscr{E}$, $n \rightarrow n_0 + \delta n$ etc, taking

$$\delta\mathscr{E} = \delta\mathscr{E}_0\,\mathrm{e}^{(\mathrm{i}kx - \omega t)} \tag{13}$$

etc. We need to look at the dependent quantities v_d, D, e_i, c and n'. For simplicity we shall assume that diffusion and capture are not affected by field. Since momentum relaxation is rapid we can assume that

$$\delta v_\mathrm{d} = \mu\delta\varepsilon \tag{14}$$

where μ is the differential mobility. The volume ionisation rate needs more care. First of all we observe the e_i is related to an ionisation cross-section σ_I via

$$e_\mathrm{i} = v_\mathrm{d}\sigma_\mathrm{i}. \tag{15}$$

Thus

$$\delta e_i = \sigma_i \delta v_d + v_d \delta \sigma_i \tag{16}$$

and δv_d depends upon local field $\delta \mathscr{E}$ whereas $\delta \sigma_i$ depends upon average field $\delta \bar{\mathscr{E}}$, where

$$\delta \bar{\mathscr{E}} = \frac{1}{L} \int_{x-L}^{x} \delta \mathscr{E} \, dx. \tag{17}$$

It is straightforward to show that

$$\delta \bar{\mathscr{E}} = \delta \mathscr{E} [(\sin kL/2)/(kL/2)] \, e^{-ikL/2}. \tag{18}$$

Also we can write

$$\delta n' = \delta n \, e^{-ikL}. \tag{19}$$

Substitution into the basic equations with the retention of linear terms only, leads to a dispersion relation which, after putting $\omega \to \omega - i\nu$ ($\nu > 0$ for stability), splits into real and imaginary parts:

$$f_1(\omega, k, \nu) = 0 \qquad f_2(\omega, k, \nu) = 0 \tag{20}$$

where

$$\begin{aligned} f_1(\omega, k, \nu) = {} & \omega^2 - \omega(kv_d + \omega_i \sin kL) - \nu^2 + \nu(\omega_T + \omega_n + \omega_c + k^2 D - \omega_i \cos kL) \\ & - \omega_n(\omega_c + k^2 D) - \omega_c[\omega_i(1 - \cos kL) + \omega_T] - \omega_n \omega_i \gamma[(\sin kL/2)/(kL/2)] \\ & \times [\cos kL/2 - (kD/v_d) \sin kL/2] \end{aligned} \tag{21}$$

$$\begin{aligned} f_2(\omega, k, \nu) = {} & \nu(kv_d - 2\omega + \omega_i \sin kL) + \omega(\omega_T + \omega_n + \omega_c + k^2 D - \omega_i \cos kL) - \omega_n k v_d \\ & + \omega_i \omega_c (1 - \sin kL) + \omega_n \omega_i \gamma[(\sin kL/2)/(kL/2)] \\ & \times [\sin kL/2 + (kD/v_d) \cos kL/2]. \end{aligned} \tag{22}$$

In these equations $\omega_I = v_d \sigma_i n_{T0}$ where σ_i is the ionisation cross-section, $\omega_T = v_d \sigma_T (N_T - n_{T_0})$ is the trapping frequency with σ_T the capture cross-section, $\omega_n = v_d(\sigma_i + \sigma_T)n$ is the generation frequency and $\omega_c = e\mu n/\varepsilon_s$ is the conductivity frequency. The quantity γ quantifies the rate by which the ionisation cross-section increases with field:

$$\gamma = \sigma_e/(\sigma_i + \sigma_T) \tag{23}$$

where

$$\sigma_e = (1/\sigma_i)(d\sigma_i/d\varepsilon) \, e/\varepsilon_s \tag{24}$$

and it is worth noting that γ is typically very large compared with unity.

The relation between the disturbances of free and trapped electron densities and the electric field are given by

$$\delta n = \frac{i\omega - \omega_c}{v_d - ikD} \frac{\varepsilon_s}{e} \delta \mathscr{E} \tag{25}$$

$$\delta n_T = ik(\varepsilon_s/e) \, \delta \mathscr{E} - \delta n. \tag{26}$$

Note that we have related volume rates to cross-sections via the drift velocity. This is entirely appropriate for the ionisation process since the ionisation rate depends upon the rate at which electrons drift in the field. For capture, however, the relevant quantity is the random thermal velocity, v_{th}. Because we do not consider explicitly any energy (and therefore field) dependence of the capture process the replacement of thermal velocity by drift velocity once the dispersion relations are derived merely affects the definition of capture cross-section. This is done purely for simplicity so that a factor of v_{th}/v_d need not be included.

4. Stationary waves

These relations give ω and ν as functions of k. They admit of many different solutions and it is necessary to focus on specific situations in which substantial simplifications can be made in order to obtain an insight. First of all, we are interested in non-oscillatory (stationary) ionisation waves, and so we can set $\omega = 0$. We shall also focus on the situation in insulators since these solids are closest to the case of a gaseous plasma in which ionisation waves are known to occur; consequently we take $\omega_c \to 0$. In wide-gap semiconductors and insulators the electric fields of interest are of the order of 10^5 to 10^7 V cm^{-1}, certainly high enough for drift velocities to reach saturation values of order 10^7 cm s^{-1}. (Note that a weak dependence of velocity on field also suggests that taking $\omega_c \to 0$ is a good approximation.) Furthermore, we expect that the wavevectors of interest will be of order $1/L$ or greater, with L of order 10^{-5} cm or less. This means that the quantity kv_d, the drift frequency, will be of order 10^{12} s^{-1}, which is large, and similarly k^2D, the diffusion frequency, will be large.

Imposing the conditions $\omega = 0$, $kv_d \gg \omega_i \sin kL$, $k^2D \gg \omega_T + \omega_n + \omega_c - \omega_i \cos kL$, leads to

$$\nu \approx k^2 D \qquad \text{or} \qquad \nu \approx \omega_n \left[1 + \frac{\omega_i \gamma}{k^2 D} \frac{\sin kL/2}{kL/2} \left(\cos kL/2 - \frac{kD}{v_d} \sin kL/2\right)\right] \tag{27}$$

and

$$\nu \approx \omega_n \left[1 - \frac{\omega_i \gamma}{k v_d} \frac{\sin kL/2}{kL/2} \left(\sin kL/2 + \frac{kD}{v_d} \cos kL/2\right)\right]. \tag{28}$$

The solution $\nu \approx k^2D$ cannot be a consistent one in general. The alternative solution can be consistent provided that

$$\frac{\sin kL/2}{k^2 D}\left(\cos kL/2 - \frac{kD}{v_d} \sin kL/2\right) = -\frac{\sin kL/2}{k v_d}\left(\sin kL/2 + \frac{kD}{v_d}\cos kL/2\right) \tag{29}$$

which reduces to

$$[1 + (kD/v_d)^2] \sin kL = 0. \tag{30}$$

This may be satisfied for the spectrum given by

$$k = s\pi/L \tag{31}$$

where $s = 0, 1, 2$ etc. Thus ionisation waves exist with this spectrum but they decay with a decay frequency given by

$$\nu = \begin{cases} \omega_n \left(1 - \dfrac{\omega_i \gamma L}{v_d} \dfrac{2}{(2r+1)^2 \pi^2}\right) & s = 2r + 1 \\ \omega_n & s = 2r \end{cases} \tag{32}$$

where r is an integer. The waves are unstable when

$$(\omega_i \gamma L/v_d)[2/(2r+1)^2\pi^2] > 1. \tag{33}$$

Only waves with s equal to an odd integer need be considered since the others always decay. A discussion of this instability is deferred to the next section.

We have shown that ionisation waves do exist in solids. The reason why they exist is closely bound up with the non-local nature of the ionisation process as quantified by L. Essentially they are waves caused by retarded action. Putting $L = 0$, which is equivalent to saying that impact ionisation is purely a local process, immediately rules out a stable or growing wave solution. The detailed character of the ionisation waves can be seen from the relation of the electron densities with the electric field, namely

$$\delta n \simeq 0 \qquad \delta n_T \simeq ik\frac{e_s}{e}\,\delta\mathscr{E} \tag{34}$$

In other words, the space charge responsible for the field oscillations resides solely in the traps. This accounts for the appearance of ω_n, the generation frequency, in the decay constant, since this determines the rate at which an excess population of trapped electrons decays. It is worth noting that trapped space charge of both signs can occur only if in the uniform state the trap is incompletely filled and not totally empty. Thus there is an implicit condition for the appearance of waves, namely

$$0 < n_{T0} < N_T. \tag{35}$$

Since the mobile electron population is not perturbed the non-local effect associated with n is not important—n can be taken to be a constant, determined by the current density, namely

$$j = en_0 v_d. \tag{36}$$

In insulators this will be determined by the field at the cathode which governs injection via, for example, Fowler–Nordheim tunnelling.

It is interesting to note that the presence of stable ionisation waves gives rise to a current-controlled negative differential resistance (NDR) under some circumstances. We can see this by taking $\nu = 0$ and considering a real sinusoidal wave: (figure 3)

$$\delta\mathscr{E} = \delta\mathscr{E}_0 \sin kx \qquad \delta\mathscr{E} = 0, x = 0. \tag{37}$$

The additional voltage generated across the sample of length W will be

$$\delta V = \int_0^W \delta\mathscr{E}\,dx = \frac{\delta\mathscr{E}_0}{k}(1 - \cos kW). \tag{38}$$

This will be added to the uniform state voltage $V_0 = \mathscr{E}_0 W$. Now, given that the current increases monotonically with $\mathscr{E}_0$, an NDR will arise if

$$d(V_0 + \delta V)/d\mathscr{E}_0 < 0 \tag{39}$$

that is

$$d\,\delta V/d\mathscr{E}_0 < -W. \tag{40}$$

Now since $k = \pi/L$ (considering the fundamental only) and $L = E_I/e\mathscr{E}_0$ we can obtain

$$\frac{d\,\delta V}{d\mathscr{E}_0} = \frac{\delta\mathscr{E}_0}{\mathscr{E}_0}\frac{L}{\pi}\left(\cos\frac{\pi W}{L} - 1 + \frac{\pi W}{L}\sin\frac{\pi W}{L}\right). \tag{41}$$

Suppose $W \gg L$. The derivative is most negative when

$$W/L = (2r + \tfrac{3}{2}). \tag{42}$$

In this case

$$d\,\delta V/d\mathscr{E}_0 \simeq -(\delta\mathscr{E}_0/\mathscr{E}_0)\,W. \tag{43}$$

and the condition for NDR becomes

$$\delta\mathscr{E}_0/\mathscr{E}_0 > 1. \tag{44}$$

This, of course, violates our condition of linearity. Nevertheless, we can see from

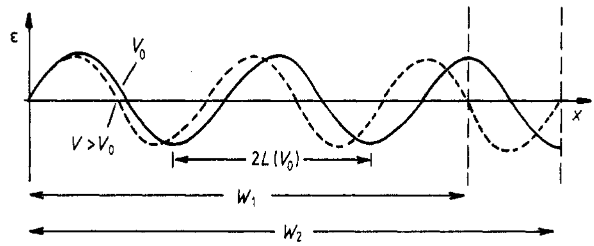

Figure 3. Stationary ionisation waves and NDR. In the case of a specimen of length W, increasing the applied voltage shortens the wavelength and increases the overall oscillatory voltage drop, and so the differential resistance is positive. In the case of a specimen of length W_2 the overall oscillatory voltage drop decreases, leading to the possibility of NDR.

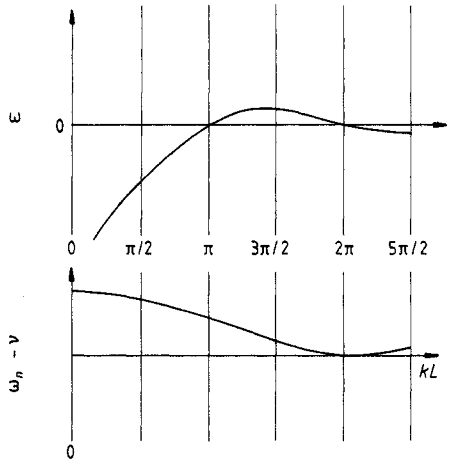

Figure 4. Dispersion relations (schematic).

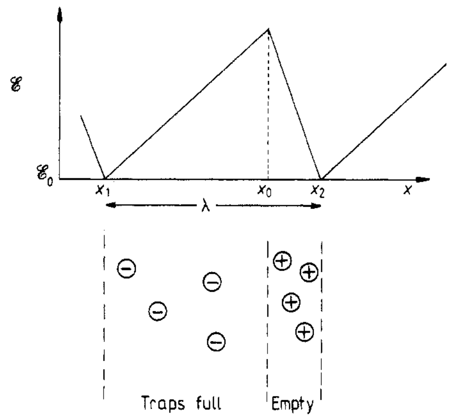

Figure 5. Saw-toothed ionisation wave.

this imperfect example that in the case of stable non-linear waves there should exist conditions on the parameter W/L under which NDR appears, a conjecture which is borne out in numerical computations (Ridley and El-Ela 1989).

5. Propagating waves

Let ω now be finite, but much smaller than the drift frequency, and maintain the assumptions $kv_d \gg \omega_T + \omega_n + \omega_c - \omega_i \cos kL$ and $\omega_c \to 0$. Then it is straightforward to show that (figure 4):

$$\omega = -\frac{\omega_n \omega_i \gamma}{kv_d} \frac{\sin kL}{kL} \qquad \omega \ll kv_d \tag{45}$$

$$\nu = \omega_n \left(1 - \frac{\omega_i \gamma_L}{2v_d} \frac{\sin^2 kL/2}{(kL/2)^2}\right). \tag{46}$$

The decay constant is unchanged in form, and this solution encompasses the stationary solution. The waves are backward travelling ($\omega < 0$) for

$$\sin kL > 0 \tag{47}$$

and forward travelling for

$$\sin kL < 0. \tag{48}$$

The group velocity v_g is related to the phase velocity v_k as follows:

$$v_g = v_k(kL \cot kL - 2). \tag{49}$$

We cannot rely on these solutions as k approaches zero since this would violate our initial assumptions. Nevertheless it is clear that the highest frequency modes are those with $kL < \pi$, and these are backward travelling. They are also potentially the ones with maximum gain. If we can put $kL \ll 1$, then

$$\omega \simeq -\omega_n \omega_i \gamma / k v_d, \tag{50}$$

$$\nu \simeq \omega_n(1 - \omega_I \gamma L / 2v_d). \tag{51}$$

These waves have a positive group velocity and grow provided that

$$\omega_i \gamma L / 2v_d > 1. \tag{52}$$

Increasing k reduces the gain, following the $[(\sin kL/2)/(kL/2)]^2$ function. A fuller treatment of the long-wavelength regime will be presented in a future report.

Another solution involving propagating waves can be found when ω is large. In this case

$$\omega = k v_d \qquad \nu = k^2 D. \tag{53}$$

These are familiar as drifting space charge waves which dissipate through diffusion. These have nothing to do with ionisation, and we do not consider them further.

We have shown that both propagating and stationary ionisation waves can exist in a solid. Propagating waves can exist in forward-travelling or backward-travelling forms. Backward waves, forward waves and stationary waves are commonly observed phenomena in weakly ionised plasmas. Here we have shown that they should also be observable in insulators. All these waves, however, are potentially unstable and it is of interest to consider the role of non-linearities in determining the end point or points of the instability.

6. Non-linear stationary waves

We shall not attempt a full discussion of the non-linear dynamics but merely point out the existence of large-amplitude stationary waves. If, as we have assumed, the free-electron density remains unperturbed and the conductivity frequency is negligible, the basic equations reduce to

$$\frac{d\mathscr{E}}{dx} = \frac{e}{\varepsilon}\left(\frac{\sigma_T N_T}{\sigma_i + \sigma_T} - n_{T0}\right). \tag{54}$$

Since σ_i depends on the fields between x and $x - L$ the equation is a difference–differential equation. A saw tooth model follows by taking $\sigma_i \gg \sigma_T$ or $\sigma_i \ll \sigma_T$. Since in this approximation all the space charge resides in the trap there exist two extreme conditions which will be achieved by growth, namely, traps empty and traps full. In such a situation the waves become approximately saw toothed in shape with the field increasing roughly as $(e/\varepsilon_s)(N_T - n_{T0})x$ when the traps are full and decreasing roughly as $-(e/\varepsilon_s)n_{T0}x$ when the traps are empty. Because of the sensitivity of the ionisation rate to field the demarcation between these states can be quite sharp. The ability of the traps to provide space charge is thus limited and this will inhibit further growth.

A simple saw tooth model is thus (figure 5):

$$\begin{aligned} x_1 \leqslant x \leqslant x_0 \qquad & \mathscr{E} = \mathscr{E}_0 + (e/\varepsilon_s)(N_T - n_{T0})(x - x_1) \\ x_0 \leqslant x < x_2 \qquad & \mathscr{E} = \mathscr{E}_0 + (e/\varepsilon_s)(N_T - n_{T0})(x_0 - x_1) - (e/\varepsilon_s)n_{T0}(x - x_0). \end{aligned} \tag{55}$$

For no net space charge $\mathscr{E} = \mathscr{E}_0$ when $x = x_1$ and $x = x_2$. The wavelength λ is thus $x_2 - x_1$. The latter is determined by the condition that the average field is equal to the critical field for ionisation at the maximum ($x = x_0$) *and* minimum ($x = x_1$ and $x = x_2$). The field rises towards its maximum in a region where trapping is dominant and the traps are full. At the maximum, ionisation dominates trapping, the traps empty and the field falls. At the minimum the average field drops below the ionisation field and trapping once more dominates.

The average field is

$$\bar{\mathscr{E}}(x) = \frac{1}{L}\left(\int_{x-L}^{x_0} \mathscr{E}\,\mathrm{d}x + \int_{x_0}^{x} \mathscr{E}\,\mathrm{d}x\right) \tag{56}$$

This turns out to be

$$\bar{\mathscr{E}}(x) - \bar{\mathscr{E}}_c = [e(x - x_0)/2\varepsilon L][(N_T - n_{T0})(x_0 - x + 2L) - n_T(x - x_0)] \tag{57}$$

where $\bar{\mathscr{E}}_c$ is the critical average field for ionisation. The equation $\bar{\mathscr{E}}(x) = \bar{\mathscr{E}}_c = 0$ has two solutions, namely,

$$x = x_0 \qquad x = x_0 + 2L(1 - n_T/N_T) = x_2. \tag{58}$$

Equating fields at $x = x_1$ and $x = x_2$ leads to $\lambda = 2L$. This means that the wavelength does not change in the non-linear regime when we assume the condition

$$\int_{x_1}^{x_2} \rho\,\mathrm{d}x = 0. \tag{59}$$

Knowing the wavelength, we can obtain the peak-to-trough amplitude which turns out to be

$$\Delta\mathscr{E} = (e/\varepsilon_s)[(N_T - n_{T0})n_{T0}/N_T]2L. \tag{60}$$

Thus, the amplitude is zero for fully empty or completely full traps and is a maximum when the traps are half filled, namely

$$\Delta\mathscr{E}_{\max} = (e/\varepsilon_s)(N_T/2)L. \tag{61}$$

As an example, taking $\varepsilon_s = 10^{-12}$ F cm^{-1}, $N_T = 10^{18}$ cm^{-3} and $L = 600$ Å, gives $\Delta\mathscr{E}_{\max} \simeq 5 \times 10^5$ V cm^{-1}.

These spatial oscillations represent a kind of limit cycle in our non-linear system. It would be interesting to investigate its stability and the subsequent evolution of the system with increasing field and increasing electron density, and work on this is in progress. Numerical work has already shown, for example, that NDR occurs under certain conditions for non-linear waves.

7. Summary

We have shown that hot electrons impact ionising and being captured by a set of traps in an insulator lead to the production of ionisation waves which can be forward travelling, stationary or backward travelling, depending on wavelength. The basic cause of their existence is the non-local nature of the ionisation process. This is quantified by the ionisation length L. The growth constant for the waves is determined by linear theory. It is shown that stationary waves in a finite sample can give rise to a current-controlled NDR. Unstable stationary waves grow into saw tooth waves with an amplitude limited by trap occupancy. The theory is applicable to all solids in which the *differential* conductivity is small and the drift velocity and diffusion is large at the relevant impact ionising fields. It therefore encompasses semiconductors as well as insulators.

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